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Gravitation and Other Field Theories

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Abstract

Relations between the Einstein theory of gravitation in curved Riemann time-space and classical field theories in flat Minkowski time-space are discussed from various points of view.

1. *Introduction*

The question has been asked as to how far the non-linear Einstein gravitation theory in curved Riemann time-space R_{1-3} (with metric $g_{\alpha\beta}$) might be recognised as a somewhat eccentric member of the family of familiar linear classical field theories in flat Minkowski time-space (with metric $\eta_{\alpha\beta}$). Some gravitation theorists regard the Einstein theory as essentially different from the flat field theories (Treder, 1971). Field theorists more engaged in flat theories seem to be predisposed to look for some kind of derivation of the Einstein gravitation theory from some flat theory (Fierz, 1939; Gupta, 1954, 1957; Thirring, 1959, 1961; Arnowitt, 1962; Feynman 1962–1963; Halpern, 1963*a*, *b*; van Nieuwenhuizen, 1973). The most cogent argument for the latter view might be expected from a fictitious bright flat theorist called Mink (perhaps to be found as far away as Feynman's Venutian representative (Feynman, 1962–1963)) who, without any advance knowledge of Einstein's general relativity theory of gravitation (though familiar with Riemann geometry), starts with a special relativistic field theory in M_{1-3} and from there makes a straightforward advance towards the Einstein theory in R_{1-3}. But such an approach appears to meet with great difficulties. We may easily understand the latter, if instead of ambitiously and vainly trying to forget the Einstein theory, we avail ourselves of the results of Einstein's creative imagination (just by comparing in a pedestrian way various aspects of the question from the flat as well as the curved point of view). It does not even demand much creativity to imagine various scenarios for inventive attempts by Mink to a genuine flat approach. Let us first see how he might start and soon would get into trouble.

2. *Flat Theory*

2.1. *'Material' Systems*

As long as one deals in flat Minkowski time-space M_{1-3} with such familiar classical (and even quantal) 'material' systems as charged point particles and the electromagnetic field, one may proceed quite a way before running into serious difficulties. In the present paper I want to deal with classical theories only. Just for simplicity I shall take classical electrodynamics as an illustrative example. The 'material' systems (point charges e_r at $\xi_r^\alpha(\tau_r)$ and electromagnetic vector field $a^\alpha(\xi)$) are, together with their interaction described by a (non-unique) Poincaré invariant Lagrangian L_m, for instance

$$L_m(\xi) = -\sum_r \int_{-\infty}^{\infty} d\tau_r\, \delta(\xi - \xi_r(\tau_r)) \left(\frac{m_r}{2}\eta_{\alpha\beta}\, \dot{\xi}_r^\alpha \dot{\xi}_r^\beta + e_r \eta_{\alpha\beta}\, \dot{\xi}_r^\alpha a^\beta\right) + \tfrac{1}{4}\eta_{\alpha\gamma}\,\eta_{\beta\delta} f^{\alpha\beta} f^{\gamma\delta} \tag{2.1.1}$$

with

$$f^{\alpha\beta} = a^{\alpha,\beta} - a^{\beta,\alpha} \tag{2.1.2}$$

By variation with regard to the particle and field quantities and to translations we obtain the (unique) equations of motion and field equations

$$\frac{\delta L_m}{\delta \xi_r^\kappa} = 0; \qquad \frac{\delta L_m}{\delta a^\kappa} = 0 \tag{2.1.3}$$

as well as the (non-unique, non-symmetric) canonical complete energy-momentum tensor $T_{m\,\mathrm{can}}^{\alpha\beta}$. The translation invariance of L_m ensures conservation of energy-momentum, expressed by the (weak) condition

$$T_{m,\lambda}^{\kappa\lambda} \underset{\mathrm{w}}{=} 0. \tag{2.1.4}$$

$T_{m\,\mathrm{can}}^{\kappa\lambda}$ may be symmetrized under preservation of (2.1.4) into for instance

$$T_{m\,\mathrm{syc}}^{\kappa\lambda}(\xi) = \sum_r \int_{-\infty}^{\infty} d\tau_r\, \delta(\xi - \xi_r(\tau_r))\, m_r\, \dot{\xi}_r^\kappa\, \dot{\xi}_r^\lambda - \eta_{\beta\delta} f^{\kappa\beta} f^{\lambda\delta} + \tfrac{1}{4}\eta^{\kappa\lambda}\,\eta_{\alpha\gamma}\,\eta_{\beta\delta} f^{\alpha\beta} f^{\gamma\delta} \tag{2.1.5}$$

The trace of this tensor is the scalar

$$\eta_{\alpha\beta}\, T_{m\,\mathrm{syc}}^{\alpha\beta}(\xi) = \sum_r \int_{-\infty}^{\infty} d\tau_r\, \delta(\xi - \xi_r(\tau_r))\, m_r\, \eta_{\alpha\beta}\, \dot{\xi}_r^\alpha\, \dot{\xi}_r^\beta \tag{2.1.6}$$

2.2. *Free Gravitation Field*

The classical flat field theories which appear the most promising candidates in Mink's approach to gravitation theory are those of a symmetric tensor (spin two) field $h_{\alpha\beta}$ coupled to the 'material' tensor $T^{\kappa\lambda}_{m\,\mathrm{syc}}$ and a scalar (spin zero) field f coupled to the 'material' scalar $\eta_{\alpha\beta}\,T^{\alpha\beta}_{m\,\mathrm{syc}}$. With a view to the asymptotic behaviour in the Newtonian limit, Mink may choose 'zero mass' right from the beginning. Since (2.1.6) would give no coupling to the electromagnetic field, he may in a first attempt take for the conjectural gravitation field just the pure tensor field $h_{\alpha\beta}$. In order to get a simple linear second-order field equation, he may for a free Lagrangian (in the absence of 'material' systems) in M_{1-3} try a Poincaré invariant scalar L_g, which should be a homogeneous and bilinear form in the first-order derivatives $h_{\alpha\beta,\gamma}$ (as will be indicated by a tilde) and equivalent to (Arnowitt, 1962)

$$\tilde{L}_g = -\frac{\kappa}{2}h_{\alpha\beta,\sigma}\,h_{\gamma\delta,\tau}\{\sigma\eta^{\alpha\gamma}\,\eta^{\beta\delta}\,\eta^{\sigma\tau} - \lambda\eta^{\alpha\beta}\,\eta^{\gamma\delta}\,\eta^{\sigma\tau} - \mu\eta^{\alpha\gamma}\,\eta^{\beta\tau}\,\eta^{\delta\sigma} - \mu\eta^{\alpha\gamma}\,\eta^{\beta\sigma}\,\eta^{\delta\tau} + 2\nu\eta^{\alpha\beta}\,\eta^{\gamma\sigma}\,\eta^{\delta\tau}\} \tag{2.2.1}$$

At some stage he would have to find criteria for choosing the real constants σ, λ, μ, ν. The variation principle with regard to $h_{\alpha\beta}$ with

$$\begin{aligned} B^{\kappa\lambda} &= -\frac{\delta\tilde{L}_g}{\delta h_{\kappa\lambda}} = -\frac{\partial\tilde{L}_g}{\partial h_{\kappa\lambda}} + \left(\frac{\partial\tilde{L}_g}{\partial h_{\kappa\lambda,\sigma}}\right)_{,\sigma} \\ &= -\kappa h_{\alpha\beta,\sigma\tau}\{\sigma\eta^{\kappa\alpha}\,\eta^{\lambda\beta}\,\eta^{\sigma\tau} - \lambda\eta^{\kappa\lambda}\,\eta^{\alpha\beta}\,\eta^{\sigma\tau} - \mu\eta^{\kappa\alpha}\,\eta^{\lambda\sigma}\,\eta^{\beta\tau} \\ &\quad - \mu\eta^{\lambda\alpha}\,\eta^{\kappa\sigma}\,\eta^{\beta\tau} + \nu\eta^{\kappa\lambda}\,\eta^{\alpha\sigma}\,\eta^{\beta\tau} + \nu\eta^{\alpha\beta}\,\eta^{\kappa\tau}\,\eta^{\lambda\sigma}\} \end{aligned} \tag{2.2.2}$$

leads to the field equation

$$B^{\kappa\lambda} = 0 \tag{2.2.3}$$

and the canonical energy-momentum tensor

$$\begin{aligned} \tilde{T}^{\kappa\lambda}_{g\,\mathrm{can}} &= \frac{\partial\tilde{L}_g}{\partial h_{\alpha\beta,\lambda}}\,h_{\alpha\beta,\delta}\,\eta^{\delta\kappa} - \eta^{\kappa\lambda}\,\tilde{L}_g \\ &= -\frac{\kappa}{2}h_{\alpha\beta,\sigma}\,h_{\gamma\delta,\tau}\{\sigma\eta^{\alpha\gamma}\,\eta^{\beta\delta}(2\eta^{\kappa\sigma}\,\eta^{\lambda\tau} - \eta^{\kappa\lambda}\,\eta^{\sigma\tau}) \\ &\quad - \lambda\eta^{\alpha\beta}\,\eta^{\gamma\delta}(2\eta^{\kappa\sigma}\,\eta^{\lambda\tau} - \eta^{\kappa\lambda}\,\eta^{\sigma\tau}) \\ &\quad - 2\mu\eta^{\alpha\gamma}(2\eta^{\lambda\beta}\,\eta^{\kappa\sigma}\,\eta^{\delta\tau} - \eta^{\kappa\lambda}\,\eta^{\delta\tau}\,\eta^{\beta\sigma}) \\ &\quad + 2\nu\eta^{\alpha\beta}(\eta^{\lambda\gamma}\,\eta^{\kappa\sigma}\,\eta^{\delta\tau} + \eta^{\lambda\gamma}\,\eta^{\kappa\tau}\,\eta^{\delta\sigma} - \eta^{\kappa\lambda}\,\eta^{\delta\tau}\,\eta^{\gamma\sigma})\} \end{aligned} \tag{2.2.4}$$

If (2.2.3) is satisfied, $\tilde{T}^{\kappa\lambda}_{g\,\mathrm{can}}$ (2.2.4) satisfies the (weak) conservation condition

$$\tilde{T}^{\kappa\lambda}_{g,\lambda} \underset{\mathrm{w}}{=} 0 \tag{2.2.5}$$

This condition remains satisfied, if one adds to (2.2.4) the ('spin type') expression

$$\{\alpha\varphi^{\kappa\lambda\sigma}+\beta\chi^{\kappa\lambda\sigma}+\gamma\psi^{\kappa\lambda\sigma}\}_{,\sigma} \tag{2.2.6}$$

because the latter satisfies (independent of the field equation) the (strong) conservation condition

$$\{\alpha\varphi^{\kappa\lambda\sigma}+\beta\chi^{\kappa\lambda\sigma}+\gamma\psi^{\kappa\lambda\sigma}\}_{,\sigma\lambda}\underset{s}{=}0 \tag{2.2.7}$$

The three superpotentials

$$\varphi^{\alpha\beta\gamma}=\frac{\kappa}{2}(\eta^{\alpha\beta}\eta^{\gamma\rho}-\eta^{\alpha\gamma}\eta^{\beta\rho})\eta^{\sigma\tau}h_{\rho\sigma,\tau} \tag{2.2.8}$$

$$\chi^{\alpha\beta\gamma}=\frac{\kappa}{2}(\eta^{\tau\beta}\eta^{\gamma\rho}-\eta^{\tau\gamma}\eta^{\beta\rho})\eta^{\sigma\alpha}h_{\rho\sigma,\tau} \tag{2.2.9}$$

$$\psi^{\alpha\beta\gamma}=\frac{\kappa}{2}(\eta^{\alpha\beta}\eta^{\gamma\tau}-\eta^{\alpha\gamma}\eta^{\beta\tau})\eta^{\rho\sigma}h_{\rho\sigma,\tau} \tag{2.2.10}$$

will appear to be the flat analogues of Wiarda's three curved superpotentials (3.2.9), (3.2.10), (3.2.11) for $n=0$.

The conserved total energy-momentum is given by the space volume integral

$$P_g^{\kappa}=\int d\sigma_0\,\tilde{T}_g^{\kappa 0} \tag{2.2.11}$$

When $h_{\alpha\beta}$ and $h_{\alpha\beta,\gamma}$ tend to zero in the limit of infinite space-like distances, the addition of (2.2.6) to (2.2.4) does not affect P_g^{κ}.

In case Mink might want to symmetrise (2.2.4) in κ, λ, perhaps after addition of (2.2.6), the only possibility appears to be

$$\mu=\nu=0 \qquad \alpha=-\beta \tag{2.2.12}$$

Another questionable criterion for choosing σ, λ, μ, ν may be suggested by the appearance of the linear gauge transformation

$$h_{\alpha\beta}\to\bar{h}_{\alpha\beta}=h_{\alpha\beta}-\Lambda_{\alpha,\beta}-\Lambda_{\beta,\alpha} \tag{2.2.13}$$

in various treatments. Λ_γ is an arbitrary gauge vector field. Unless Λ_γ is infinitesimal, it seems problematic whether one might or even should not add in the last member of (2.2.13) a term bilinear in $\Lambda_{\gamma,\delta}$. Invariance of $B^{\kappa\lambda}$ (2.2.2) under the gauge transformation would require

$$\lambda=\mu=\nu \tag{2.2.14}$$

Invariance of $\tilde{L}_g$ (2.2.1) and $\tilde{T}_g^{\kappa\lambda}$ (2.2.4) cannot be attained at all. That need not be too serious, since they are not unique anyhow.

As a further step Mink might perhaps (for convenience or for deeper field theoretical reasons) try to restrict $\bar{h}_{\alpha\beta}$ to the special Hilbert gauge (Thirring, 1959) for which

$$\eta^{\alpha\beta}\bar{h}_{\alpha\beta}=0 \qquad \eta^{\alpha\gamma}\bar{h}_{\alpha\beta,\gamma}=0 \qquad \eta^{\gamma\delta}\bar{h}_{\alpha\beta,\gamma\delta}=0 \tag{2.2.15}$$

by imposing on the finite vector Λ_γ the conditions

$$\eta^{\alpha\beta}\Lambda_{\alpha,\beta}=\tfrac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta} \tag{2.2.16}$$

$$\eta^{\alpha\gamma}\Lambda_{\beta,\alpha\gamma}=\eta^{\alpha\gamma}h_{\alpha\beta,\gamma}-\tfrac{1}{2}\eta^{\alpha\gamma}h_{\alpha\gamma,\beta} \tag{2.2.17}$$

These conditions are only consistent if $h_{\alpha\beta}$ satisfies the field equation (2.2.3) with the choice

$$\sigma=\lambda=\mu=\nu \tag{2.2.18}$$

If with this choice he starts in this special gauge for which the subsidiary conditions (2.2.15) are satisfied with a new Lagrangian

$$\bar{L}_g=-\frac{\kappa}{2}\bar{h}_{\alpha\beta,\sigma}\bar{h}_{\gamma\delta,\tau}\,\sigma\{\eta^{\alpha\gamma}\eta^{\beta\delta}\eta^{\sigma\tau}-\eta^{\alpha\gamma}\eta^{\beta\tau}\eta^{\delta\sigma}\} \tag{2.2.19}$$

he gets analogous to (2.2.2) and (2.2.4)

$$\bar{B}^{\kappa\lambda}=-\kappa\bar{h}_{\alpha\beta,\sigma\tau}\,\sigma\{\eta^{\kappa\alpha}\eta^{\lambda\beta}\eta^{\sigma\tau}-\eta^{\kappa\alpha}\eta^{\lambda\sigma}\eta^{\beta\tau}\}=0 \tag{2.2.20}$$

and a new energy-momentum tensor

$$\bar{T}_g^{\kappa\lambda}=\frac{\kappa}{2}\bar{h}_{\alpha\beta,\sigma}\bar{h}_{\gamma\delta,\tau}\,\sigma\eta^{\alpha\gamma}\eta^{\beta\delta}(2\eta^{\kappa\sigma}\eta^{\lambda\tau}-\eta^{\kappa\lambda}\eta^{\sigma\tau}) \tag{2.2.21}$$

The field equation (2.2.3) is then identically fulfilled, but its role is taken over by the subsidiary conditions (2.2.15). In this gauge the total energy becomes

$$\bar{P}_g^0=\int d\sigma_0\,\bar{T}_g^{00}=\frac{\kappa}{2}\sigma\int d\sigma_0\{\bar{h}_{\alpha\beta,0}\bar{h}_{\gamma\delta,0}+\bar{h}_{\alpha\beta,k}\bar{h}_{\gamma\delta,l}\eta^{kl}\}\eta^{\alpha\gamma}\eta^{\beta\delta} \tag{2.2.22}$$

where latin indices are space-like. If Mink finally requires that this $\bar{P}_g^0$ shall share the property of a total free field energy that it may not be negative that fixes the sign of σ and (2.2.18) becomes

$$\sigma=\lambda=\mu=\nu>0 \tag{2.2.23}$$

This choice determines the sign of $B^{\kappa\lambda}$ and of the classical free field propagator in M_{1-3}. If the theory does lead to a proper gravitation theory indeed, this sign will ultimately decide in the Newtonian limit for attraction against repulsion of masses (Thirring, 1959).

The choice (2.2.18) is a sufficient and also necessary condition for the validity of

$$B^{\kappa\lambda}{}_{,\lambda}\underset{s}{=}0 \tag{2.2.24}$$

independent of the field equation. Still that does not make $B^{\kappa\lambda}$ a serious candidate for an energy-momentum tensor.

The symmetry condition (2.2.12) is incompatible with the choice (2.2.18) So far none of the arguments for a choice of σ, λ, μ, v appears coercive. Mink must be clever enough to think of other and better ones. In quantum field theory van Nieuwenhuizen has derived the choice (2.2.23) from the condition that ghosts and tachyons must be excluded (van Nieuwenhuizen, 1973). It would be interesting to derive corresponding arguments from classical flat free tensor field theory.

2.3. *Coupling of Gravitation and Matter*

As long as Mink deals exclusively with the gravitation field, he does not run a great risk of meeting with inconsistencies, but the theory remains void of physical meaning. He is liable to hit on inconsistencies as soon as he tries to couple in M_{1-3} the symmetric tensor field $h_{\alpha\beta}$ to the symmetric 'external' 'material' (that is non-gravitational) energy-momentum tensor $T_m^{\alpha\beta}$ of all other present particles and fields. If the latter are restricted to our familiar classical 'material' systems, the complete Lagrangian becomes

$$L(\xi) = L_g(\xi) + L_m(\xi) + L_i(\xi) \tag{2.3.1}$$

with the Poincaré invariant interaction term

$$L_i = -\tau\kappa\, h_{\alpha\beta} T_m^{\alpha\beta}{}_{\text{syc}} \tag{2.3.2}$$

The real factor τ to the coupling constant κ still remains to be chosen. The 'material' equations of motion and field equations under the influence of the $h_{\alpha\beta}$ field then follow from variation of the part

$$L_m + L_i = L_m - \tau\kappa\, h_{\alpha\beta} T_m^{\alpha\beta}{}_{\text{syc}} \tag{2.3.3}$$

Variation of the part $L_g + L_i$ of (2.3.1) with regard to $h_{\alpha\beta}$ leads to the gravitation field equation

$$B^{\kappa\lambda} = -\tau\kappa\, T_m^{\kappa\lambda}{}_{\text{syc}} \tag{2.3.4}$$

$T_m^{\kappa\lambda}$ appears as the 'external' source, which generates the $h_{\alpha\beta}$ field as described by the generator $B^{\kappa\lambda}$, which is still given by (2.2.2). Because the free Lagrangian $\tilde{L}_g$ (2.2.1) has been confined to a homogeneous and bilinear form in the first-order derivatives $h_{\alpha\beta,\gamma}$, the generator (2.2.2) is homogeneous and linear in the second-order derivatives $h_{\alpha\beta,\gamma\delta}$. The field equation (2.3.4) has then to be supplemented by suitable boundary conditions on the field $h_{\alpha\beta}$ and its first-order derivatives $h_{\alpha\beta,\gamma}$.

The translation invariance of the complete Lagrangian (2.3.1) now ensures conservation of the complete energy-momentum $T^{\alpha\beta}$

$$T^{\kappa\lambda}{}_{,\lambda} \underset{\text{w}}{=} 0 \tag{2.3.5}$$

In the complete energy-momentum tensor

$$T^{\alpha\beta} = T_g^{\alpha\beta} + T_m^{\alpha\beta} + T_i^{\alpha\beta} \tag{2.3.6}$$

the interaction part has to be derived from L_i by variation with regard to all 'internal' and 'external' variables. But since $T_m^{\alpha\beta}$ in (2.3.2) is independent of $h_{\alpha\beta}$, the 'internal' variation gives no contribution to $T_i^{\alpha\beta}$. The 'external' variation of the part (2.3.3) then leads to the part

$$T_m^{\alpha\beta} + T_i^{\alpha\beta} \tag{2.3.7}$$

(2.3.3) and (2.3.7) might so to say be considered as the complete 'material' parts in the (conjectural) gravitation field $h_{\alpha\beta}$. Owing to exchange between the three terms in the second member of (2.3.6), the partial conditions (2.1.4) and (2.2.5) remain in general not satisfied under the interaction. The field equation (2.3.4) is therefore not compatible with the condition (2.2.24), that is with the choice (2.2.18). At first sight that might suggest Mink that (2.2.18) is a wrong choice.

If we turn to the curved point of view, we shall be able to see in an easy and clear way, that so far the flat theory of the conjectural gravitation field $h_{\alpha\beta}$ coupled to the 'material' energy-momentum tensor is inconsistent for any choice of σ, λ, μ, ν and that the choice (2.2.23) is just the appropriate one, which may help our friend Mink out of his difficulties.

3. *Curved Theory*

3.1. *Einstein Theory*

In the Einstein theory the gravitation field is (somewhat implicitly) represented in the metric tensor $g_{\alpha\beta}$. The latter has to obey the non-linear second-order gravitation equation

$$G^{\kappa\lambda} = -\kappa\, T_m^{\kappa\lambda} \tag{3.1.1}$$

together with suitable boundary conditions. The generator

$$G^{\kappa\lambda} = R^{\kappa\lambda} - \tfrac{1}{2} g^{\kappa\lambda} R \tag{3.1.2}$$

of the gravitation field is a functional with terms either linear in $g_{\alpha\beta,\gamma\delta}$ or bilinear in $g_{\alpha\beta,\gamma}$ with various (respectively three or four) factors $g^{\alpha\beta}$. The source $T_m^{\alpha\beta}$ represents again a complete 'external' symmetric 'material' energy-momentum tensor of all other particles and fields. But now it has to be expressed as a contravariant tensor in the curved Riemann coordinates. As the covariant divergence of $G^{\kappa\lambda}$ (3.1.2) equals zero, also $T_m^{\kappa\lambda}$ has to satisfy the condition

$$T_m{}^{\kappa\lambda}{}_{;\lambda} = T_m{}^{\kappa\lambda}{}_{,\lambda} + T_m^{\alpha\lambda}\Gamma_\alpha{}^\kappa{}_\lambda + T_m^{\kappa\beta}\Gamma_\beta{}^\lambda{}_\lambda \underset{\mathrm{W}}{=} 0 \tag{3.1.3}$$

The 'material' equations of motion and field equations (and the condition (3.1.3)) as well as the gravitation equation (3.1.1) may be derived by variation from a (non-unique) Lagrangian density $\sqrt{(-g)}L$ with

$$L(x) = L_g(x) + L_m(x) \tag{3.1.4}$$

The idea is that the interaction between gravitation field and the 'material' systems is entirely accounted for by the metric tensor $g_{\alpha\beta}$ (and perhaps its derivatives) in L_m, so that there is no need for an interaction term L_i such as in (2.3.1) (where (2.3.3) might be considered as the complete 'material' part). In order to yield the field equation (3.1.1), the 'internal' variation with regard to $g_{\alpha\beta}$ has to give (apart from a common factor)

$$\frac{\delta(\sqrt{(-g)}\,L_g)}{\delta g_{\kappa\lambda}} = -\frac{1}{2\kappa}\sqrt{(-g)}\,G^{\kappa\lambda} \tag{3.1.5}$$

and

$$\frac{\delta(\sqrt{(-g)}\,L_m)}{\delta g_{\kappa\lambda}} = -\tfrac{1}{2}\sqrt{(-g)}\,T_{m\ \mathrm{met}}^{\kappa\lambda} \tag{3.1.6}$$

L_g is a functional of $g_{\alpha\beta}$, $g_{\alpha\beta,\gamma}$ and $g_{\alpha\beta,\gamma\delta}$ only. But L_m also contains 'material' tensors. The functional derivative (3.1.6) of the latter is only defined (weakly) uniquely if the action integral of $\sqrt{(-g)}\,L_m$ is stationary with respect to infinitesimal variations of the 'material' particle and field variables, that is if the 'material' equations of motion and field equations are satisfied.

If we require the free gravitation Lagrangian L_g to be an invariant scalar, it is given by

$$L_g = -\frac{1}{2\kappa}R \tag{3.1.7}$$

The terms of the expression for the curvature R are either linear in $g_{\alpha\beta,\gamma\delta}$ or bilinear in $g_{\alpha\beta,\gamma}$, all with various (respectively two or three) factors $g^{\alpha\beta}$. One may also choose (Schmutzer, 1968)

$$\tilde{L}_g = \frac{1}{2\kappa}\tilde{R} \tag{3.1.8}$$

where $\tilde{R}$ is a (non-invariant) part of R, which is homogeneous and bilinear in $g_{\alpha\beta,\gamma}$ (as indicated by a tilde) and reads

$$\tilde{R} = -\tfrac{1}{4}\,g_{\alpha\beta,\sigma}\,g_{\gamma\delta,\tau}\{g^{\alpha\gamma}g^{\beta\delta}g^{\sigma\tau} - g^{\alpha\beta}g^{\gamma\delta}g^{\sigma\tau} - g^{\alpha\gamma}g^{\beta\tau}g^{\delta\sigma} - g^{\alpha\gamma}g^{\beta\sigma}g^{\delta\tau} + 2g^{\alpha\beta}g^{\gamma\sigma}g^{\delta\tau}\} \tag{3.1.9}$$

For the electrodynamic example the density of the 'material' Lagrangian L_m becomes

$$\sqrt{(-g)}\,L_m(x) = -\sum_r \int_{-\infty}^{\infty} d\tau_r\,\delta(x - x_r(\tau_r))\left(\frac{m_r}{2}g_{\alpha\beta}\,\dot{x}_r^\alpha\dot{x}_r^\beta + e_r\,g_{\alpha\beta}\,\dot{x}_r^\alpha a^\beta\right) + \tfrac{1}{4}\sqrt{(-g)}\,g_{\alpha\gamma}\,g_{\beta\delta}f^{\alpha\beta}f^{\gamma\delta} \tag{3.1.10}$$

where the contravariant tensor $f^{\alpha\beta}$ may still be written in the form (2.1.2). (3.1.10) contains no derivatives of $g_{\alpha\beta}$. In the particle terms the density $\sqrt{-g}$ has been absorbed in the four-dimensional δ-function. The density of the metric energy-momentum tensor derived from (3.1.6) becomes

$$\sqrt{(-g)}\, T_m^{k\lambda}{}_{\text{met}} = \sum_r \int_{-\infty}^{\infty} d\tau_r\, \delta(x - x_r(\tau_r))\, m_r\, \dot{x}_r^\kappa\, \dot{x}_r^\lambda$$
$$- \sqrt{(-g)}\, g_{\beta\delta} f^{\kappa\beta} f^{\lambda\delta} + \tfrac{1}{4}\sqrt{(-g)}\, g^{\kappa\lambda} g_{\alpha\gamma} g_{\beta\delta} f^{\alpha\beta} f^{\gamma\delta} \qquad (3.1.11)$$

which is equivalent to the density of the symmetrised canonical energy-momentum tensor (2.1.5) written in curved coordinates. A general investigation in how far the metric 'material' energy-momentum tensor is equivalent to the symmetrised canonical one would be desirable.

3.2. *Complete Source Conservation*

The present Section 3.2 may well be passed over until Section 5.3.

In order to get in the curved Einstein theory a generally conserved energy-momentum, one usually adds to the 'external' 'material' symmetric (metric) energy-momentum tensor $T_m^{\alpha\beta}$ (3.1.6) some kind of 'internal' gravitation energy-momentum complex $T_g^{\alpha\beta}$ (as a functional of $g^{\alpha\beta}$, $g_{\alpha\beta,\gamma}$ and $g_{\alpha\beta,\gamma\delta}$ only), so that

$$T^{\alpha\beta} = T_m^{\alpha\beta} + T_g^{\alpha\beta} \qquad (3.2.1)$$

obeys the conservation condition

$$(\sqrt{(-g)}^n\, T^{\kappa\lambda})_{,\lambda} \underset{\text{w}}{=} 0 \qquad (3.2.2)$$

in which the weight n of the conserved density still has to be chosen. The most natural choice would seem $n = 1$. Since the condition (3.2.2) is non-contravariant, $T^{\alpha\beta}$ and therefore also $T_g^{\alpha\beta}$ could only be contravariant tensors in the trivial case that $T^{\alpha\beta}$ would vanish identically. That may be achieved by the choice $n = 1$ together with the metric definition of $T_g^{\alpha\beta}{}_{\text{met}}$ by

$$\frac{\delta(\sqrt{(-g)}\, L_g)}{\delta g_{\kappa\lambda}} = -\tfrac{1}{2}\sqrt{(-g)}\, T_g^{k\lambda}{}_{\text{met}} \qquad (3.2.3)$$

This results with (3.1.5), (3.1.1) (or directly from variation of (3.1.4)) in the (weak) cancellation of complete energy-momentum

$$T_g^{k\lambda}{}_{\text{met}} + T_m^{k\lambda}{}_{\text{met}} \underset{\text{w}}{=} 0 \qquad (3.2.4)$$

That removes, together with all difficulties, all physical interest as well.

The fact that $T_g^{\alpha\beta}$ is not only non-unique (just as $T_m^{\alpha\beta}$), but (in the non-trivial case) even non-contravariant, makes proper localisation of energy-momentum by a density-current density entirely delusive. One may at best define an overall energy-momentum, for instance integrated (over a space-like cross-section of R_{1-3}) over insular systems.

If one could find one or more functionals $A^{\alpha\beta}$ of $g^{\alpha\beta}$, $g_{\alpha\beta,\gamma}$ and $g_{\alpha\beta,\gamma\delta}$ only, which satisfy independent of the field equation (3.1.1) the strong conservation condition of weight n

$$(\sqrt{(-g)}^n\, A^{\kappa\lambda})_{,\lambda} \underset{\text{s}}{=} 0 \qquad (3.2.5)$$

one might split the generator $G^{\alpha\beta}$ (3.1.2) into two (non-contravariant) functionals

$$G^{\alpha\beta} = A^{\alpha\beta} + \kappa T_g^{\alpha\beta} \tag{3.2.6}$$

The contravariant Einstein equation (3.1.1) may then be written in the crypto-contravariant form

$$A^{\kappa\lambda} = -\kappa\, T^{\kappa\lambda} \tag{3.2.7}$$

The complete source term, given by (3.2.1) satisfies the (weak) conservation condition (3.2.2). It seems a matter of taste as to how far one is willing to accept $A^{\alpha\beta}$ as some kind of generalised generator with respect to a complete 'external' plus 'internal' source. In this particular interpretation of the non-linear curved theory it appears as an essential difference from familiar linear flat theories, that the conserved complete source contains an 'internal' part $T_g^{\alpha\beta}$, which need not vanish in the free field case of a vanishing 'external' part.

It might perhaps appear more satisfactory if the functionals $A^{\alpha\beta}$ and $T_g^{\alpha\beta}$ would be derived from a Lagrangian form. But we shall soon see that this is not always possible.

For a functional $A^{\alpha\beta}$ of $g^{\alpha\beta}$, $g_{\alpha\beta,\gamma}$ and $g_{\alpha\beta,\gamma\delta}$ satisfying the strong condition (3.2.5) Wiarda has derived the general form (Wiarda, 1973)

$$A^{\alpha\beta} = \sqrt{(-g)}^{-n}(\alpha\, \Phi_n^{\alpha\beta\gamma} + \beta\, X_n^{\alpha\beta\gamma} + \gamma\, \Psi_n^{\alpha\beta\gamma})_{,\gamma} \tag{3.2.8}$$

The superpotentials Φ, X and Ψ are afine tensor densities of weight n, antisymmetric in β,γ and determined by

$$\Phi_n^{\alpha\beta\gamma} = \tfrac{1}{2}\sqrt{(-g)}^n(g^{\alpha\beta}g^{\gamma\rho} - g^{\alpha\gamma}g^{\beta\rho})g^{\sigma\tau}g_{\rho\sigma,\tau} \tag{3.2.9}$$

$$X_n^{\alpha\beta\gamma} = \tfrac{1}{2}\sqrt{(-g)}^n(g^{\tau\beta}g^{\gamma\rho} - g^{\tau\gamma}g^{\beta\rho})g^{\sigma\alpha}g_{\rho\sigma,\tau} \tag{3.2.10}$$

$$\Psi_n^{\alpha\beta\gamma} = \tfrac{1}{2}\sqrt{(-g)}^n(g^{\alpha\beta}g^{\gamma\tau} - g^{\alpha\gamma}g^{\beta\tau})\, g^{\rho\sigma}g_{\rho\sigma,\tau} \tag{3.2.11}$$

If $A^{\alpha\beta}$ is required to be symmetric in α, β, the coefficients α, β, γ have to satisfy the condition

$$-\alpha = \beta = \frac{2}{n}\gamma \tag{3.2.12}$$

Just as $G^{\alpha\beta}$ in (3.2.6), $A^{\alpha\beta}$ and hence also $T_g^{\alpha\beta}$ contain terms either linear in $g_{\alpha\beta,\gamma\delta}$ or bilinear in $g_{\alpha\beta,\gamma}$ with various (respectively three and four) factors $g^{\alpha\beta}$. $\tilde{T}_g^{\alpha\beta}$ is homogeneous and bilinear in $g_{\alpha\beta,\gamma}$ under the condition

$$-\alpha = \beta = \gamma = 1 \tag{3.2.13}$$

If one wants such a $\tilde{T}_g^{\alpha\beta}$ also to be symmetric both (3.2.12) and (3.2.13) have to be satisfied, in which case $n = 2$. This is just the Landau choice (Wiarda, 1973).

It has been remarked by Wiarda that, in order to obtain a functional $A_\alpha{}^\beta$ satisfying the corresponding divergence condition

$$(\sqrt{(-g)}^n A_\kappa{}^\lambda)_{,\lambda} = 0 \tag{3.2.14}$$

one has to lower the index α in (3.2.8) within the brackets in the second member.

In case a $\tilde{T}_{g\alpha}{}^{\beta}$ (for which (3.2.13) is satisfied) may be derived from a Lagrangian $\tilde{L}_g$ by the canonical relation

$$\sqrt{(-g)}\,\tilde{T}_{g\kappa}{}^{\lambda} = \frac{\partial(\sqrt{(-g)}\,\tilde{L}_g)}{\partial g_{\alpha\beta,\lambda}}\, g_{\alpha\beta,\kappa} - g_{\kappa}{}^{\lambda}\sqrt{(-g)}\,\tilde{L}_g \qquad (3.2.15)$$

it satisfies the relation

$$\tilde{L}_g = -\tfrac{1}{2} g^{\alpha}{}_{\beta}\, \tilde{T}_{g\alpha}{}^{\beta} \qquad (3.2.16)$$

If the second member does not yield a Lagrangian, $\tilde{T}_{g\alpha}{}^{\beta}$ (and hence also $A_{\alpha}{}^{\beta}$) cannot be canonically derived from another Lagrangian either.

4. *Confrontation*

4.1. *Projection*

For an easy comparison between a curved description in R_{1-3} and a flat one in M_{1-3}, I shall imagine the curved Riemann time-space R_{1-3} to be embedded in a flat Minkowski hyper-time-space M_{t-s} with sufficiently large numbers t and s of time- and space-like dimensions. It seems dubious whether such an embedding is in general mathematically possible at all. But in fact I shall use it only as a somewhat sloppy way of speaking, which in a more rigorous mathematical treatment should be replaced by an abstract mapping between R_{1-3} and M_{1-3} (Treder, 1971). Terms like 'projection' and 'tangent' may even be taken over from the embedding picture in the abstract mapping.

In M_{t-s} we choose an arbitrary flat M_{1-3} and also linear orthonormal Minkowski coordinates (ξ^{α},ξ^{a}), such that M_{1-3} is characterized by the equation

$$\xi^{a} = 0 \qquad (4.1.1)$$

for the additional coordinates ξ^{a}, whereas ξ^{α} represent internal Minkowski coordinates in that M_{1-3}. For the sake of simplicity, we may transform the x-coordinate system in R_{1-3} into an a-system, by which I shall mean, that the projection of a time-space point x^{α} in this system in R_{1-3} on the chosen M_{1-3} is given by

$$\xi^{\alpha} = x^{\alpha} \qquad (4.1.2)$$

The projection of an a-coordinate net is then an ortholinear Minkowski net. The subgroup of curved x-coordinate transformations between a-systems is projected on the Poincaré group of transformations in all M_{1-3}'s embedded in M_{t-s}. The equation of R_{1-3} in M_{t-s} may be written

$$\xi^{a} = f^{a}(\xi^{\beta}) \qquad (4.1.3)$$

The internal Riemann metric $g_{\alpha\beta}$ in R_{1-3} in this a-system then becomes

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \eta_{ab}\frac{\partial f^{a}}{\partial \xi^{\alpha}}\frac{\partial f^{b}}{\partial \xi^{\beta}} \qquad (4.1.4)$$

The determinant of the matrix $g_{\alpha\beta}$ is g. If the subdeterminant of the element $g_{\alpha\beta}$ is $\bar{G}^{\beta\alpha}$, $g^{\alpha\beta}$ becomes

$$g^{\alpha\beta} = g^{-1}\, \bar{G}^{\alpha\beta} \tag{4.1.5}$$

with determinant g^{-1}. (4.1.2), (4.1.4) and (4.1.5) are tricky hybrid expressions. The indices of x^α and $g_{\alpha\beta}$ are in R_{1-3} lowered and raised with the help of $g_{\alpha\beta}$ and $g^{\alpha\beta}$, those of ξ^α and $\eta_{\alpha\beta}$ in M_{1-3} with $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$.

The problem of embedding of R_{1-3} in M_{t-s} is whether (under certain integrability conditions) for a given metric $g_{\alpha\beta}$ in R_{1-3} it is possible to find for sufficiently large t and s one or more solutions $f^a(\xi^\beta)$ of (4.1.4). If not, we may as well forget (4.1.3) and (4.1.4) after having written down (4.1.8).

Now we confront the curved Einstein theory in R_{1-3} with Mink's flat theory in M_{1-3}. Let us first suppose that for his free gravitation field he has already made the choice (2.2.18). Then there is a striking correspondence between ${}_c\tilde{L}_g$ (3.1.8), (3.1.9) and ${}_fL_g$ (2.2.1)

$${}_c\tilde{L}_g \leftrightarrow \frac{\omega^2}{4\sigma}\, {}_f\tilde{L}_g \tag{4.1.6}$$

for

$$g^{\alpha\beta} \leftrightarrow \eta^{\alpha\beta} \qquad g_{\alpha\beta,\gamma} \leftrightarrow \omega\kappa\, h_{\alpha\beta,\gamma} \tag{4.1.7}$$

That suggests that one writes (4.1.4) and (4.1.5) as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \omega\kappa\, h_{\alpha\beta} \tag{4.1.8}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} + \omega\kappa\, k^{\alpha\beta} \tag{4.1.9}$$

and to consider the second member of (4.1.6) as the lowest order approximation of a series expansion in powers of the small gravitation coupling constant κ. This is quite a tricky business, since in (4.1.8), (4.1.9) $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are co- and contravariant components of a tensor in the curved metric, $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$ similarly in the flat metric and $h_{\alpha\beta}$ and $k^{\alpha\beta}$ do not represent components of the same tensor at all.

If we confront ${}_cL_m$ and ${}_fL_m + {}_fL_i$, we get in zero order

$${}_cL_m \leftrightarrow {}_fL_m \tag{4.1.10}$$

Consistency with (4.1.6) requires

$$\omega^2 = 4\sigma \tag{4.1.11}$$

For the confrontation of the first-order terms we need the (tricky) correspondence for functional derivatives of the type

$$\frac{\delta(\sqrt{(-g)}^n\, {}_cL)}{\delta g_{\kappa\lambda}} \leftrightarrow \frac{\partial_f L}{\partial \eta_{\kappa\lambda}} + \frac{n}{2}\eta^{\kappa\lambda}\, {}_fL + \frac{1}{\omega\kappa}\frac{\delta_f L}{\delta h_{\kappa\lambda}} \tag{4.1.12}$$

That leads for weight $n = 1$ with (3.1.6) and (2.3.2) to

$$\omega\kappa\, h_{\alpha\beta}\frac{\delta(\sqrt{(-g)}\, {}_cL_m)}{\delta g_{\alpha\beta}} = -\tfrac{1}{2}\omega\kappa\, h_{\alpha\beta}\, {}_cT^{\alpha\beta}_{m\ \mathrm{met}} \leftrightarrow {}_fL_i = -\tau\kappa\, h_{\alpha\beta}\, {}_fT^{\alpha\beta}_m \tag{4.1.13}$$

The last correspondence is exact, if we take also in the flat theory, notably in the interaction term (2.3.2) the metric energy-momentum tensor ${}_fT_m^{\alpha\beta}$. In those cases where ${}_cL_m$ does not contain derivatives of $g_{\alpha\beta}$, the latter is according to the correspondence (4.1.12) with $n=1$ defined by

$$\frac{\partial\ {}_fL_m}{\partial\eta_{\kappa\lambda}}+\tfrac{1}{2}\eta^{\kappa\lambda}\ {}_fL_m=-\tfrac{1}{2}\ {}_fT^{\kappa\lambda}_{m\ \mathrm{met}} \tag{4.1.14}$$

A consistency condition for the correspondence (4.1.13) would together with (4.1.11) be

$$\omega=2\tau \tag{4.1.15}$$

For the gravitation Lagrangian $\tilde{L}_g$ the correspondence (4.1.12) would with (3.1.5) and (2.2.2) for (4.1.11) lead to

$$-\frac{1}{2\kappa}\sqrt{(-g)}\,G^{\kappa\lambda}\leftrightarrow\frac{\partial_f\tilde{L}_g}{\partial\eta_{\kappa\lambda}}+\tfrac{1}{2}\,\eta^{\kappa\lambda}\,{}_f\tilde{L}_g-\frac{1}{\omega\kappa}\,B^{\kappa\lambda} \tag{4.1.16}$$

From the curved point of view the first two (extra) terms in the second member may already cast some doubt on the validity of the variation (2.2.2).

In some cosmological models the curvature tensor goes to zero in the limit of infinite space-like distances. Then in any a-system $g^{\alpha\beta}$ tends to $\eta^{\alpha\beta}$ in this limit and $g_{\alpha\beta,\gamma}$ to zero. For such models it might be sensible to choose M_{1-3} asymptotic tangential to R_{1-3} in this limit. In some other models the projection of the whole R_{1-3} on a M_{1-3} may be restricted to a part of the latter. For the moment we need not bother about cosmology. It is sufficient to consider a suitably restricted region of R_{1-3} and its projection on a suitably chosen M_{1-3}.

4.2. *Local Inertial System*

Let us for a while restrict ourselves even to the infinitesimal surroundings of a time-space point x_0 in R_{1-3} and choose M_{1-3} tangential to R_{1-3} at this point. (Infinitesimal is meant with respect to all separate components δx^α of a displacement in the coordinate system in question.) Then we get for the a-system in R_{1-3} a local inertial a-system, which in this normal neighbourhood is a normal coordinate system (Nomizu, 1956) and satisfies at x_0 the local boundary conditions

$$g^{\alpha\beta}=\eta^{\alpha\beta}\qquad g_{\alpha\beta,\gamma}=0\,(\Gamma_{\alpha\ \beta}^{\ \gamma}=0);\qquad g_{\alpha\beta,\gamma\delta}=-\eta_{\alpha\kappa}\eta_{\beta\lambda}g^{\kappa\lambda}{}_{,\gamma\delta} \tag{4.2.1}$$

It represents so to say the time-space surroundings of a freely falling non-rotating material system at x_0. In the tangent M_{1-3} we get at the corresponding ξ_0 the boundary conditions

$$h_{\alpha\beta}=0=k^{\alpha\beta};\qquad h_{\alpha\beta,\gamma}=0=k^{\alpha\beta}{}_{,\gamma};\qquad h_{\alpha\beta,\gamma\delta}=-\eta_{\alpha\kappa}\eta_{\beta\lambda}k^{\kappa\lambda}{}_{,\gamma\delta} \tag{4.2.2}$$

with at the point of contact the correspondence

$$g_{\alpha\beta,\gamma\delta} = \omega\kappa\, h_{\alpha\beta,\gamma\delta} \tag{4.2.3}$$

and in its surroundings

$$-g \approx \exp(\omega\kappa\, h_{\sigma\tau}\,\eta^{\sigma\tau}) \tag{4.2.4}$$

As long as we shall restrict local correspondence relations to the lowest order terms in the infinitesimal surroundings of the point of contact between R_{1-3} and M_{1-3}, the hybrid tensor qualities of (4.1.8), (4.1.9) will not yet become effective and with sufficient care inconsistencies of correspondence may be avoided.

A standard procedure to derive a curved description of 'material' systems in the presence of a gravitation field from a flat Minkowski description in the absence of gravitation, is to start with this flat description in the infinitesimal surroundings of the point of contact ξ_0 in the local tangent M_{1-3} at x_0. Owing to (4.2.2) the 'material' systems then appear locally not subjected to gravitation action. The same description still holds in the infinitesimal surroundings of x_0 in the corresponding local inertial a-system in R_{1-3} (as experienced from a freely falling position). Then it may directly be translated into a covariant description in an arbitrary curved coordinate system in these local surroundings and finally it may be extended (owing to the covariance) to the whole curved R_{1-3}. From the curved point of view the flat description of 'material' systems is correct in the whole M_{1-3} in the absence of a gravitation field and the curved description is correct in the whole R_{1-3} in the presence of a gravitation field corresponding to the metric $g_{\alpha\beta}$.

It is therefore perhaps not too surprising that, whereas a flat theory of the free conjectural gravitation tensor field $h_{\alpha\beta}$ in the absence of 'matter' may also be consistent over the whole M_{1-3}, it appears difficult to wrench a theory of coupled gravitation and 'material' systems into M_{1-3}. From the curved point of view it seems almost obvious to confront the curved and flat theories in the infinitesimal surroundings of the point of contact $x_0 = \xi_0$ in R_{1-3} and M_{1-3}, that is under the local boundary conditions (4.2.1), (4.2.2) with the correspondence (4.2.3), (4.2.4).

Under these conditions both ${}_c\tilde{L}_g$ (3.1.8), (3.1.9) and ${}_f\tilde{L}_g$ (2.2.1) locally become zero at $x_0 = \xi_0$. Their infinitesimal variations correspond as

$$\delta(\surd(-g)\,{}_c\tilde{L}_g) \leftrightarrow \frac{\omega^2}{4\sigma}\,\delta_f\tilde{L}_g \tag{4.2.5}$$

The generator (3.1.2) tends locally at x_0 to

$$G^{\kappa\lambda} = \tfrac{1}{2}g_{\alpha\beta,\sigma\tau}\{\eta^{\alpha\sigma}(\eta^{\kappa\lambda}\,\eta^{\beta\tau} - \eta^{\kappa\beta}\,\eta^{\lambda\tau}) + \eta^{\kappa\alpha}(\eta^{\lambda\beta}\,\eta^{\sigma\tau} - \eta^{\lambda\sigma}\,\eta^{\beta\tau}) - \eta^{\alpha\beta}(\eta^{\kappa\lambda}\,\eta^{\sigma\tau} - \eta^{\kappa\sigma}\,\eta^{\lambda\tau})\} \tag{4.2.6}$$

The tricky first two (extra) terms in the second member of (4.1.16) tend locally to zero. The remaining straightforward correspondence

$$G^{\kappa\lambda} \leftrightarrow \frac{2}{\omega} B^{\kappa\lambda} \tag{4.2.7}$$

is consistent with (4.2.6) and (2.2.2) for

$$\sigma = 1 \tag{4.2.8}$$

(4.2.5) is still consistent with the zero-order correspondence (4.1.10) for (4.1.11) and with the first-order correspondence (4.1.13), (4.1.14) for (4.1.15). Collecting the consistency conditions (4.1.11), (4.1.15), (4.2.8) we have

$$\sigma = \tau = 1 \qquad \omega = 2 \tag{4.2.9}$$

The local limit of (3.1.3) at x_0 becomes for (4.2.1)

$$T^{\kappa\lambda}_{m\ ,\lambda} \underset{\mathrm{w}}{=} 0 \tag{4.2.10}$$

The 'material' energy-momentum appears locally conserved by itself in a local inertial system, because from such a position there appears no gravitational action on the local 'matter'. With (4.2.2) the condition (4.2.10) will also be locally satisfied in the flat theory. With the flat field equation (2.3.4) this is now locally consistent with the condition (2.2.24), that is with the choice (2.2.23), on which all our correspondence relations for the gravitation quantities are based.

The curved theory is considered to be consistent over all (or at least a finite part of) R_{1-3}. The flat theory appears only locally consistent in M_{1-3} in the infinitesimal surroundings of a point of contact $\xi_0 = x_0$ with local boundary conditions (4.2.2), (4.2.1). Under these conditions the variation in (2.2.2) is also acceptable from a curved point of view and (4.2.10) becomes compatible with (2.2.24). With the local correspondence (4.2.1), (4.2.2) and the choice (4.2.9) the flat and curved theories are locally equivalent in these surroundings.

In studying relations between flat and curved theories, we may attain better understanding from the local tangential correspondence in a local inertial system than from the inconsistent and tricky projection correspondence (4.1.8), (4.1.9). But for poor Mink this point of view is not readily accessible and we still have to see in which ways he might proceed.

4.3. *Energy-Momentum in Local Inertial System*

Together with Section 3.2 the present Section 4.3 may well be passed over until Section 5.3.

In a local inertial a-system with (4.2.1) at x_0, where the generator (3.1.2) reads in the local limit as (4.2.6), $A^{\kappa\lambda}$ (3.2.8) and $T_g^{\kappa\lambda}$ in (3.2.6) read in the local limit

$$A^{\kappa\lambda} = \tfrac{1}{2}\, g_{\alpha\beta,\sigma\tau}\{\alpha\eta^{\alpha\sigma}(\eta^{\kappa\lambda}\eta^{\beta\tau} - \eta^{\kappa\beta}\eta^{\lambda\tau}) - \beta\eta^{\kappa\alpha}(\eta^{\lambda\beta}\eta^{\sigma\tau} - \eta^{\lambda\sigma}\eta^{\beta\tau}) - \gamma\,\eta^{\alpha\beta}(\eta^{\kappa\lambda}\eta^{\sigma\tau} - \eta^{\kappa\sigma}\eta^{\lambda\tau})\} \tag{4.3.1}$$

$$T_g^{\kappa\lambda} = \tfrac{1}{2}\, g_{\alpha\beta,\sigma\tau}\{(1-\alpha)\,\eta^{\alpha\sigma}(\eta^{\kappa\lambda}\eta^{\beta\tau} - \eta^{\kappa\beta}\eta^{\lambda\tau}) + (1+\beta)\,\eta^{\kappa\alpha}(\eta^{\lambda\beta}\eta^{\sigma\tau} - \eta^{\lambda\sigma}\eta^{\beta\tau}) - (1-\gamma)\,\eta^{\alpha\beta}(\eta^{\kappa\lambda}\eta^{\sigma\tau} - \eta^{\kappa\sigma}\eta^{\lambda\tau})\} \tag{4.3.2}$$

In this representation $A^{\kappa\lambda}$ and $T_g^{\kappa\lambda}$ are already symmetric in κ, λ for $\alpha = -\beta$ (corresponding to the last condition in (2.2.12)). With (3.2.13) $\tilde{T}_g^{\kappa\lambda}$ vanishes locally entirely in the local inertial system. But so far this has no special significance. In a local inertial system (with vanishing gravitational action) the local vanishing of the divergence $T_g^{\kappa\lambda}{}_{,\lambda}$ and not the vanishing of $T_g^{\kappa\lambda}$ itself is essential.

If one transforms from the local inertial system to a general coordinate system in R_{1-3}, the transformation of (4.2.6) is unique under the condition that it has to be contravariant. The transformation of (4.3.1) and (4.3.2) is only unique under the non-contravariant condition that (3.2.5) and 3.2.6) must be preserved. In fact (4.2.6) and (4.3.1) can for the choice (3.2.13) not even be distinguished in the local inertial system.

5. *Flat Scenarios*

In our advantageous position from which both the curved and flat points of view are accessible, we can easily read the correspondence relations in both directions. The direction from curved to flat is perhaps easier than the opposite one. Poor Mink has merely his inconsitent flat theory and his brains. For us it is easy to see what kind of brilliant ideas he might need, in order to remove the inconsistencies and to invent ultimately from that flat point of view the curved theory.

Let us imagine three scenarios. None of them will be unknown, at least for physicists who cannot honestly forget the Einstein theory. The first scenario (metric substitution) is based on the tricky and misleading projection correspondence of Section 4.1, and needs some trial and error. The second one (free fall base) is based on the local tangential correspondence in a local inertial system of Section 4.2. It is straightforward and consistent all the time, but it requires more creative inventiveness indeed. The third one (perturbation expansion) has been tried by various field theorists.

5.1. *Metric Substitution*

Our correspondence relations between curved and flat theories might, for Mink, sooner or later appear as substitutions, read in the direction from flat to curved description. As to the projection correspondence of Section 4.1. his determinative step should be the substitution (4.1.6) with (4.1.7). But from where might he get the inventive idea to substitute the curved metric $g_{\alpha\beta}$ (4.1.8), (4.1.9) for the flat metric $\eta_{\alpha\beta}$? An obvious suggestion might come from the 'material' equations of motion and field equations, which he derives from the 'material' part (2.3.3) of the flat Lagrangian L by variation with regard to the 'material' variables. If he applies them for instance to clocks and measuring rods, he will find almost the same behaviour as we would find up to first order in κ in a curved R_{1-3} with metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 2\tau\kappa\, h_{\alpha\beta} \tag{5.1.1}$$

In a scrupulous elaboration it might require quite a lot of trial and error to adjust proper co- (and contra-)variance after the start from the hybrid form (5.1.1), and to extend the series expansion in powers of κ. But as soon as Mink conjectures the fundamental importance of $g_{\alpha\beta}$ as a new metric in a curved R_{1-3} and recognises after the substitution (4.1.6) the new ${}_c\tilde{L}_g$ (3.1.8), (3.1.9) as the correct free gravitation Lagrangian in this R_{1-3}, he may in a more adventurous mood go straight to the point, without bothering too much about a tricky correspondence with an inconsistent flat theory. He might make the substitution of $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$ directly in the free 'material' flat Lagrangian ${}_fL_m$ (2.1.1) and in the meantime forget ${}_fL_i$ (2.3.2) as being only the first-order term in this substitution.

The suggestion of the substitution of $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$ (5.1.1) does not strictly demand a detailed study of the behaviour of clocks and measuring rods or other specific 'material' systems. If for 'material' systems for which ${}_cL_m$ does not contain derivatives of $g_{\alpha\beta}$ in the curved description, Mink might have discovered the flat metric energy-momentum tensor ${}_fT^{\alpha\beta}_{m\,\text{met}}$ of (4.1.14), he could write (2.3.3) as

$$L_m + L_i = L_m + 2\tau\kappa\, h_{\alpha\beta}\frac{\partial L_m}{\partial \eta_{\alpha\beta}} + \tau\kappa\, h_{\alpha\beta}\,\eta^{\alpha\beta} L_m \tag{5.1.2}$$

That might suggest to regard the second member as the lowest (zero and first) order terms of a series expansion in powers of κ for the substitution of $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$ (5.1.1) in the Lagrangian density $\surd(-g)L_m$.

In Section 5.3 we shall look into the question whether a series expansion in powers of κ of a free gravitation Lagrangian density $\surd(-g)^n L_g$ of weight n might be useful for a flat scenario.

5.2. *Free Fall Base*

The second scenario requires from Mink perhaps an inventivity and creative imagination comparable to Einstein's genius. It would pay, anyhow, since after some three imaginative daring initial steps, the whole approach is straightforward and elegant and it is consistent all the time. Even if we have no hope ever to find a Mink, this ambitious scenario might perhaps elucidate some relations between flat and curved theory much better than could the other two scenarios.

The second-order field equation (2.3.4) needs supplementary zero and first-order boundary conditions. As a first step we let Mink discover that he can make his flat theory of coupled gravitation and 'matter' at least locally consistent at an arbitrary (maybe his own) time-space point ξ_0 in M_{1-3} by imposing the local boundary conditions (4.2.2) and restricting the theory for the time being to the infinitesimal surroundings of ξ_0. To be sure, that would extinguish all gravitational action on the local 'material' systems.

It would be a further adventurous and crucial step to conjecture that the adequate local reference frame corresponding to this fantastic idea would be found in the infinitesimal surroundings of a freely falling (accelerated) and non-rotating laboratory. Once having taken office in such unusual surroundings, he will occupy a supreme bridge-head for further progress.

For the third step Mink should get the bright idea of transforming from the flat M_{1-3} to a locally tangent curved R_{1-3}. We let him first consider a finite region around his local time-space point ξ_0. He may find that locally freely falling non-rotating frames at different time-space points ξ (with in general a relative acceleration) may be connected by integration of infinitesimal Poincaré transformations. One way to a curved metric might be found by the imaginative idea of considering the manifold of flat locally freely falling reference systems at the various points ξ as envelope of a curved R_{1-3}. The space of tangent M_{1-3}'s is then a dual R^*_{1-3}, which may be mapped on the R_{1-3} of corresponding time-space points of contact x with the help of reference tetrads (Treder, 1971; Møller, 1966). If the situation at all time-space points ξ should be described equally (for every point from a position in one of its own local free fall reference bases), the conjectural gravitational field $h_{\alpha\beta}$ would (almost paradoxically) have to be transformed to zero at all these points. That might be achieved by means of integration of infinitesimal gauge transformations (2.2.13). The locally consistent gravitational field equation (2.3.4) and 'material' equations of motion and field equations derived from (2.3.3) should all by a suitable coordinate countertransformation from M_{1-3} to R_{1-3} be locally preserved at $x_0 = \xi_0$ and at the same time made consistent in all the other points. As soon as Mink has got the idea of transforming from M_{1-3} to a locally tangent R_{1-3}, he needs in the first instance only to perform such a transformation in the infinitesimal surroundings of ξ_0. For this he has to find (perhaps by arguments similar to those in the first scenario) that the appropriate local infinitesimal transformation is determined by the local boundary conditions (4.2.1), (4.2.2) and the local correspondence (4.2.3), (4.2.4). Then he may readily derive the local correspondence relation of Section 4.2 read in the direction from flat to curved.

After these three imaginative tentative steps Mink might proceed straightforward in an elegant consistent way, by inventing the standard procedure by which in a curved theory a flat description of 'material' systems is usually adapted to the curved metric in R_{1-3} by starting from a local inertial system. First he might transform in the infinitesimal surroundings of x_0 (where his theory is locally consistent) all expressions and equations from the normal a-system into a general coordinate system. Then he will have to extend these expressions and equations from the infinitesimal surroundings of x_0 over the whole (or at least a finite part of) R_{1-3}. A necessary and sufficient condition that this extension shall be possible and consistent, is that the relevant expressions and equations have been transformed into an essentially co- (or contra-) variant form. (They may be crypto-covariant, that is equivalent to a covariant form.)

In the case Mink had not already made the choice (2.2.18), he would find that this is a necessary and sufficient condition for transforming $B^{\kappa\lambda}$ (2.2.2) into a contravariant form. Just as Einstein, when he had to choose the generator (3.1.2) as a contravariant tensor functional of $g_{\alpha\beta}$ and its first- and second-order derivatives, almost had not other choice. Nevertheless it might give some satisfaction if the choice (2.2.23) could already be made on sound arguments in the (not yet manifestly inconsistent) free flat gravitation theory of Section 2.2.

We may easily read the details of the procedure in the last part of the scenario from the local tangential correspondence relations in Section 4.2. Apart from the choice (2.2.23) Mink will sooner or later have to make the choice (4.2.9). Then the extended theory over R_{1-3} will finally be equivalent with the Einstein theory indeed. That means that the ultimate criterion for acceptability of the theory by observational testing whenever that may be feasible, is just the same as for Einstein's theory.

In discussing a flat approach from a curved point of view, it seems justified to ask for the roots of curvature and non-linearity by which gravitation theory is distinguished from other familiar field theories. As long as they are not coupled together, familiar 'material' systems and the free gravitation field $h_{\alpha\beta}$ (with $h_{\alpha\beta}$ considered as a proper flat tensor) both can be described in M_{1-3} in a (within certain limits) simple consistent or quasi-consistent way. As soon as they are coupled, a simple consistent description (with $h_{\alpha\beta}$ as a proper tensor) is hardly possible in M_{1-3}, whereas it is possible in an elegant and natural way in R_{1-3}. In the infinitesimal surroundings of a local inertial system (freely falling laboratory) with local boundary conditions (4.2.2), (4.2.1) and local tangential correspondence (4.2.3), (4.2.4) we can readily see how in the second-order field equation (2.3.4) for $h_{\alpha\beta}$ or the local limit of (3.1.1) for $g_{\alpha\beta}$ the presence of a 'material' energy-momentum source $T_m^{\alpha\beta}$ induces a genuine curvature in the metric $g_{\alpha\beta}$, which at the same time represents the gravitation field. In this way so to speak the present 'matter' induces a bending of the M_{1-3} of the free gravitation field into a curved R_{1-3}. But at the same time the gravitation field in R_{1-3} so to say drags along the M_{1-3} of the 'material' systems, rolling it over the induced curvature of R_{1-3} and generating the dual envelope R^*_{1-3}. Or in a more unified picture the description of the 'material' systems may also be transformed from the dual R^*_{1-3} to the R_{1-3} of the gravitation field. This wrenching of the 'material' systems into the curved R_{1-3} is in a manner of speaking a feedback effect of gravitation on 'matter' from the curvature of the gravitation field induced by the 'material' systems themselves.

The non-linearity of the curved gravitation theory is for instance manifest in the fact that the gravitation Lagrangian (3.1.8) or perhaps (3.1.7) consists of terms bilinear in $g_{\alpha\beta,\gamma}$ and perhaps also terms linear in $g_{\alpha\beta,\gamma\delta}$ with various (respectively three or two) factors $g^{\alpha\beta}$. Or that the curved generator (3.1.2) consists of terms linear in $g_{\alpha\beta,\gamma\delta}$ and other terms bilinear in $g_{\alpha\beta,\gamma}$ with a number (respectively three or four) of factors $g^{\alpha\beta}$. Along the lines of our second scenario (as well as in the first one) the predestination to this property may in the free flat gravitation theory of Section 2.2 already be read off from the corresponding property that the free flat gravitation Lagrangian (2.2.1) consists of terms bilinear in $h_{\alpha\beta,\gamma}$ with various (in fact three) factors $\eta^{\alpha\beta}$ or that the flat generator (2.2.2) consists of terms linear in $h_{\alpha\beta,\gamma\delta}$ with a number (in fact three) of factors $\eta^{\alpha\beta}$. From this special point of view the roots of non-linearity spring so to speak equally from all these factors $\eta^{\alpha\beta}$.

5.3. *Perturbation Expansion*

It is sometimes suggested that the root of non-linearity lies in a coupling of the gravitation field with its own 'internal' energy-momentum. In as far as the Einstein equation (3.1.1) is considered as correct, it accounts explicitly only for a coupling with the 'external' 'material' energy-momentum source $T_m^{\alpha\beta}$ in the second member. Any kind of self-coupling should be implied already in the non-linearity of the generator in the first member. The alternative crypto-contravariant form (3.2.7) of (3.1.1) suggests in a somewhat questionable way a coupling with a somewhat dubious conserved complete energy-momentum in terms of a somewhat dubious alternative generator.

In the inconsistent flat theory of Section 2.3 there appears no indication of any explicit or implicit coupling of the conjectural gravitation field $h_{\alpha\beta}$ with its own 'internal' energy-momentum. That has prompted various attempts to make the theory consistent and perhaps to produce the non-linearity and even the curvature of the Einstein theory by adding to the coupling of the field $h_{\alpha\beta}$ with the 'external' energy-momentum source also a coupling with the 'internal' one. In various cases this has been attempted in terms of some perturbation expansion. The underlying idea is that the expansion is in powers of the small gravitation coupling constant κ and that the inconsistent flat field equation (2.3.4) with the generator $B^{\kappa\lambda}$ (2.2.2) is the lowest order approximation to the correct field equation. Usually σ, λ, μ, ν are at some stage for some or other reason already chosen as in (2.2.23). We know already that and why that appears sooner or later as the appropriate choice.

In those cases where the explicit cosmological asymptotic assumption is made, that the flat metric $\eta_{\alpha\beta}$ will in all orders remain the correct metric in the limit of infinite space-like distances, it is obvious that whenever a curved R_{1-3} will emanate from the successive approximations, it will be asymptotically tangential to the M_{1-3} of the initial flat theory. Without such an assumption the relation between the initial M_{1-3} and a possible final R_{1-3} might be as loose as in the projection correspondence of our first scenario. If it were as tight as in the local tangential correspondence in our second scenario, the perturbation expansion would be almost superfluous. It might then at best be a poor substitute for the standard procedure of contravariant extension in the last part of the free fall scenario.

The general perturbation procedure is that in every successive step one derives from the last-order term of the gravitation Lagrangian a corresponding order contribution to the 'internal' gravitation energy-momentum. If this contribution is coupled (with strength constant $\tau\kappa$) to the gravitation field $h_{\alpha\beta}$, it gives the next order contribution to the self-interaction part of the complete gravitation Lagrangian. A flat third scenario based on this idea would be much nearer to our first scenario than to our second one. The merits of the perturbation approach would be optimal if the series expansion of the gravitation Lagrangian would represent the appropriate substitution of the flat metric $\eta_{\alpha\beta}$ by the curved metric $g_{\alpha\beta}$, so that the idea of transforming from M_{1-3} to R_{1-3} would be reached in a pedestrian way. The most conclusive success of this kind of perturbation approach would be gained if it would directly yield a crypto-contravariant field equation (3.2.7), which clearly exhibits a coupling to some complete energy-momentum source. But the generator $A^{\kappa\lambda}$ in (3.2.7) may not be derivable from a Lagrangian and it would be conceivable that already in the course of the procedure terms would have been shifted between the two members of the field equation. The approach might still be convincing if it were to produce directly the contravariant field equation (3.1.1).

In fact I am not aware of any treatment in which a closed expression is derived for the sum of the perturbation series, or even for the general νth order term. At best one sometimes argues what the result of the infinite sum should be. If one requires that the resulting field equation (as one has learned from Einstein) should be contravariant in R_{1-3}, there is hardly any uncertainty left and one is (like Einstein was) almost exclusively committed to equation (3.1.1) with the generator (3.1.2). (Just as ultimately one is almost exclusively committed to the flat choice (2.2.18).) In a genuine flat scenario that argument presupposes in the first place that our Mink would already on other grounds have conceived the idea to transform the metric $\eta_{\alpha\beta}$ in M_{1-3} into a metric $g_{\alpha\beta}$ in R_{1-3} (and to counter-transform $h_{\alpha\beta}$ to zero). That would already take away a great deal of the merits of the perturbation approach, so that Mink would hardly need it any more. In the second place the argument presupposes that the purturbation expansion correctly accounts for the gravitation self-coupling and that it is consistent and convergent. I shall try to argue that this presupposition is open to doubt. It is not at all unusual that false arguments put us on the track of ultimately (within certain boundaries) correct theories, but even that is hardly happening here. Rather than that the argument clarifies the relations between flat and curved theories, it needs a clarification from such relations.

I shall not try to analyse existing perturbation expansions. As illustration of reasons for doubt and possible pitfalls I shall discuss a simple-minded version of a part of the third flat scenario. (Much of the rest has already been discussed in the first one.) Let us in the series expansion in powers of κ write (2.3.1) as

$$L = L_m^{(0)} + \sum_{\nu=1}^{\infty} L_i^{(\nu)} + \sum_{\nu=1}^{\infty} \tilde{L}_g^{(\nu)} \tag{5.3.1}$$

where the upper indices in brackets on L_m in (2.1.1) and in the series expansions of L_i and $\tilde{L}_g$ denote the power of κ. We take the lowest order term $L_i^{(1)}$ identical with L_i (2.3.2) and $\tilde{L}_g^{(1)}$ with $\tilde{L}_g$ (2.2.1). The role of the expansion of the 'material' part $L_m + L_i$ has already been considered in the first scenario in Section 5.1 and for the moment we shall only be concerned with the expansion of $\tilde{L}_g$. We define $B^{(\nu)\kappa\lambda}$ and $\tilde{T}_g^{(\nu)\kappa\lambda}$ analogous to the relations between the first and second members of (2.2.2) and (2.2.4). The last members give then just the lowest order terms $B^{(1)\kappa\lambda}$ and $\tilde{T}_g^{(1)\kappa\lambda}$ respectively. The main point of the scenario is the conjecture that, analogous to the 'material' interaction term (2.3.2), the contribution of the νth order 'internal' energy-momentum term $\tilde{T}_g^{(\nu)\kappa\lambda}$ to the $(\nu + 1)$th order self-interaction term $\tilde{L}_g^{(\nu+1)}$ is

$$\tilde{L}_g^{(\nu+1)} = -\tau\kappa\, h_{\alpha\beta}\, \tilde{T}_g^{(\nu)\alpha\beta} \tag{5.3.2}$$

With the relations analogous to (2.2.4) this may be written as

$$\tilde{L}_g^{(\nu+1)} = -\tau\kappa \frac{\partial \tilde{L}^{(\nu)}}{\partial h_{\kappa\lambda,\beta}} h_{\kappa\lambda,\gamma}\, \eta^{\gamma\alpha}\, h_{\alpha\beta} + \tau\kappa \tilde{L}_g^{(\nu)} \eta^{\alpha\beta}\, h_{\alpha\beta} \tag{5.3.3}$$

With the initial term $\tilde{L}_g^{(1)}$ we obtain for the sum of the series expansion of $\tilde{L}_g$ the functional equation

$$(1 - \tau\kappa h_{\alpha\beta}\, \eta^{\alpha\beta})\tilde{L}_g + \tau\kappa \frac{\partial \tilde{L}_g}{\partial h_{\kappa\lambda,\beta}} h_{\kappa\lambda,\gamma}\, \eta^{\gamma\alpha}\, h_{\alpha\beta} = \tilde{L}_g^{(1)} \tag{5.3.4}$$

which after all we might have written down immediately without the help of a series expansion. It might be good luck for Mink if he would not succeed in solving this equation (5.3.4), because the result would seem to be wrong and liable to lead him astray.

We known that with the choice (4.2.9) the corresponding ${}_c\tilde{L}_g$ and ${}_f\tilde{L}_g$ in (4.1.6) are transformed into each other by the substitutions (4.1.7). The first of these substitutions or its inverse may be represented by the operator

$$\exp\left(-2\kappa\, k^{\alpha\beta} \frac{\partial}{\partial g^{\alpha\beta}}\right) \quad \text{or} \quad \exp\left(2\kappa\, k^{\alpha\beta} \frac{\partial}{\partial \eta^{\alpha\beta}}\right) \tag{5.3.5}$$

and the second one or its inverse by

$$\exp\left\{(2\kappa h_{\alpha\beta,\gamma} - g_{\alpha\beta,\gamma}) \frac{\partial}{\partial g_{\alpha\beta,\gamma}}\right\}$$

or

$$\exp\left\{(g_{\alpha\beta,\gamma} - 2\kappa h_{\alpha\beta,\gamma}) \frac{\partial}{2\kappa \partial h_{\alpha\beta,\gamma}}\right\} \tag{5.3.6}$$

respectively. If these substitution operators operate on ${}_c\tilde{L}_g$ (3.1.8), (3.1.9) or on ${}_f\tilde{L}_g$ (2.2.1) and if they are expanded in powers of κ, only the terms up to third order of (5.3.5) and those up to second order of (5.3.6) are effective. If ${}_c\tilde{L}_g$ is partially transformed by the first substitution operator of (5.3.6) into ${}_p\tilde{L}_g$, the density $\sqrt{(-g)^n}\,{}_p\tilde{L}_g$ of the latter is further transformed by the first substitution of (5.3.5) into ${}_f\tilde{L}_g$ as

$$\exp(-n\kappa h_{\alpha\beta}\eta^{\alpha\beta})\sum_{\nu=0}^{3}\frac{1}{\nu!}\left(-2\kappa\, k^{\alpha\beta}\frac{\partial}{\partial g^{\alpha\beta}}\right)(\sqrt{(-g)^n}\,{}_p\tilde{L}_g)$$

$$= \exp(-n\,\kappa\, h_{\alpha\beta}\,\eta^{\alpha\beta})\frac{1}{2}\frac{\partial(\sqrt{(-g)^n}\,{}_p\tilde{L}_g)}{\partial h_{\kappa\lambda,\sigma}}\, h_{\mu\nu,\tau}\,\eta^{\mu\alpha}\,\eta^{\nu\beta}\,\eta^{\tau\gamma}$$

$$\cdot\{\eta_{\kappa\alpha}\,\eta_{\lambda\beta}\,\eta_{\sigma\gamma} + 2\kappa(\eta_{\kappa\alpha}\,\eta_{\lambda\beta}\,h_{\sigma\gamma} + \eta_{\kappa\alpha}\,h_{\lambda\beta}\,\eta_{\sigma\gamma} + h_{\kappa\alpha}\,\eta_{\lambda\beta}\,\eta_{\sigma\gamma})$$

$$+ 4\kappa^2(h_{\kappa\alpha}\,h_{\lambda\beta}\,\eta_{\sigma\gamma} + h_{\kappa\alpha}\,\eta_{\lambda\beta}\,h_{\sigma\gamma} + \eta_{\kappa\alpha}\,h_{\lambda\beta}\,h_{\sigma\gamma})$$

$$+ 8\kappa^3\, h_{\kappa\alpha}\,h_{\lambda\beta}\,h_{\sigma\gamma}\} = {}_f\tilde{L}_g \qquad (5.3.7)$$

Of course this is an awkward way of representing the simple substitution (5.3.5). But it may show the shortcomings of the functional equation (5.3.4). We have already taken $\tilde{L}_g^{(1)}$ identical with ${}_f\tilde{L}_g$. Let us be willing to read $\tilde{L}_g$ in (5.3.4) as $\sqrt{(-g)^n}\,{}_p\tilde{L}_g$ with $n = 1$ (or perhaps $n = 0$) and condone that (5.3.4) does not explicitly account of the substitution (5.3.6). If we expand the second member of (5.3.7) in powers of κ, the first member of (5.3.4) represents the whole zero-order term, only some of the first-order terms and no higher order terms at all. Even if we do not worry too much that the density exponential has been approximated (for weight $n = 1$ or perhaps even $n = 0$), the neglect of all terms other than $\eta_{\kappa\alpha}\eta_{\lambda\beta}\eta_{\sigma\gamma}$ and $2\kappa\eta_{\kappa\alpha}\eta_{\lambda\beta}h_{\sigma\gamma}$ within the curly brackets of (5.3.7) appears a rather serious shortcoming indeed. We have concluded at the end of Section 5.2 that the roots of non-linearity (and consequently of gravitation self-interaction) spring equally from *all* the three factors in $\tilde{L}_g$ (2.2.1) or $B^{\kappa\lambda}$ (2.2.2). In this simple-minded version of the perturbation approach the self-coupling to the 'internal' energy-momentum may at best account for one of these three factors. I do not see that any other reasonable definition of $T_g^{(\nu)\alpha\beta}$ in terms of $\tilde{L}_g^{(\nu)}$ could lead to the full functional equation (5.3.7). Even if Mink, misled by (4.1.14), might get the dubious idea of regarding the expression

$$\frac{\partial_f\tilde{L}_g}{\partial\eta_{\kappa\lambda}} + \tfrac{1}{2}\eta^{\kappa\lambda}\,{}_f\tilde{L}_g = -\tfrac{1}{2}\,{}_f\tilde{T}_g^{\kappa\lambda}{}_{\mathrm{im}} \qquad (5.3.8)$$

as defining some kind of (from the point of view of (3.2.3) and (4.1.16) truncated or improper flat) metric 'internal' energy-momentum ${}_f\tilde{T}_{g\,\mathrm{im}}^{\kappa\lambda}$, that would with (5.3.2) and $\omega = 2\tau$ only lead him to the functional equation

$$(1 - \tau\kappa\, h_{\alpha\beta}\,\eta^{\alpha\beta})\tilde{L}_g - 2\tau\kappa\, h_{\kappa\lambda}\frac{\partial\tilde{L}_g}{\partial\eta_{\kappa\lambda}} = \tilde{L}_g^{(1)} \qquad (5.3.9)$$

(The splitting of the second member of (4.1.16) for $G^{\kappa\lambda}$ in (3.1.5) or for $T^{\kappa\lambda}_{g\,\mathrm{met}}$ in (3.2.3) into two parts corresponding to ${}_f\tilde{T}^{\kappa\lambda}_{g\,\mathrm{im}}$ in (5.3.8) and $B^{\kappa\lambda}$ (2.2.2) appears somewhat similar to the splitting (3.2.6).) The addition of $\tau\kappa_f\tilde{T}^{\kappa\lambda}_g$ in the first member of the field equation (2.3.4) removes the incompatibility with (2.2.24) and (2.2.18).) In (5.3.9) all factors $\eta_{\alpha\beta}$ are now treated equally, but not yet completely, since (5.3.9) only represents the terms in (5.3.7) of zero and first order in κ. (This may readily be understood from the point of view of the projection correspondence, since the addition of the correction term (5.3.8) in (4.1.16) only contributes a first-order correction in the field equation (2.3.4).) I am afraid that also in better versions of the perturbation expansion than the present simple-minded one, the self-coupling to the 'internal' energy-momentum may play a part, but not all of the game.† It seems that the non-linearity represents a deeper-rooted kind of self-interaction.

This might also be a warning to be rather careful if one might attempt to define and construct classical symmetric (or even quantal causal and anti-causal) gravitation propagators by means of a flat perturbation approach.

6. *Conclusion*

Since Einstein has conceived his gravitation theory in R_{1-3} in his own original way, independent of other familiar field theories in (also his own) M_{1-3}, it is quite easy to survey from an epigonic curved point of view relations between flat theories in M_{1-3} and curved theories in R_{1-3}. But we should perhaps assign to our good flat Mink a genius comparable to that of Einstein in order to find the latter's theory independently, from a veritable flat approach. All the same, if he never would succeed in presenting us the same theory, which Einstein already has given us from another approach, we might learn a bit from his difficulties. It is not only valuable that we may learn from looking at the same theory from various points of view. It sometimes happens that alternative representations are equivalent in the established form of a theory, but that some of them do allow for certain generalisations or other modifications, whereas some others do not.

There appears no striking likeness between the curved Einstein gravitation theory in R_{1-3} and other familiar flat 'material' theories in M_{1-3}. Nevertheless there is a close local tangential kinship, which one might experience in the infinitesimal surroundings as seen from a freely falling non-rotating laboratory. But as soon as one tries to extend the theories farther over time-space, they appear to behave quite disparately. Without gravitational action, the other familiar 'material' theories would have a

† *Note added in the proof.* This difficulty might seem to be evaded in a procedure by Deser (Deser, 1970) in which only one of the three factors $\eta^{\alpha\beta}$ appears explicitly. The other two are implicitly accounted for in the (upper index of) two factors $\Gamma^{\alpha}_{\kappa\lambda}$. Deser is an expert also in the curved theory, but for Mink it would seem a very difficult task indeed to hit from his flat point of view only on such a refined procedure in an appropriate form.

consistent natural extension in flat M_{1-3}. The theory of a gravitation field generated by the presence of 'matter' has an elegant consistent natural extension in a tangent curved R_{1-3}. By feedback the local tangential M_{1-3} reference frames for describing the 'material' systems are dragged along so to say by the gravitation field in R_{1-3}, rolling over its curved hypersurface so as to form the tangential envelope of R_{1-3} as a dual R^*_{1-3}. In a more unified representation the 'material' systems become under the gravitational (re)action wrenched at measure in the R_{1-3}. Of course all this does not exclude equivalent, though less elegant and less transparent representations in terms of mappings (for instance by projection) on a M_{1-3}.

Some indications of the inconsistency of a simple extension of the flat theory of interacting gravitation and 'matter' beyond the infinitesimal surroundings in a local inertial system are for instance the hybrid tensor character of $h_{\alpha\beta}$ and $k^{\alpha\beta}$ in the correspondence (4.1.8), (4.1.9) and the truncated form (2.2.2) of variation with regard to gravitation quantities instead of such a form as (4.1.12). It seems somewhat misleading that a flat free gravitation theory in the unphysical case of absence of 'matter' as in Section 2.2 may be extended over the whole M_{1-3} without inconsistencies, because in that case the shortcomings do not yet become manifest.

It appears that our notions of time and space in common life and imagination hardly reach beyond our infinitesimal time-space surroundings and that all extension over either M_{1-3} or R_{1-3} is a matter of more or less specialised reasoning. Within such surroundings there appears a close relationship between gravitation theory and familiar 'material' theories if they are regarded from the unusual point of view of a freely falling non-rotating observer. The answer to our opening question, as to how far the Einstein theory may be recognised as a perhaps complicated and twisted member of the family of the other familiar field theories, seems more or less a matter of taste without serious consequences.

A more interesting question is which tensor and spinor boson and fermion theories are 'familiar' or perhaps 'material' in the sense that (in the absence of other systems) they fit consistently in flat M_{1-3} of special relativity (even when they interact with each other) and which other ones (like some tensor and scalar theories) demand a R_{1-3} or perhaps some other non-Riemann time-space, at least when they interact with, for instance, familiar 'material' systems of the former type.

Owing to the subtlety of the relations between the flat and curved points of view, the pro- and contra-qualities of the approach from the one or the other side (Thirring, 1961) and the consequences of their mutual relations demands a rather scrupulous discussion. For instance the most persuasive argument that the flat approach can decide about the attractive or repulsive character of gravitation interaction, whereas the curved approach could not, deserves a critical examination from either point of view. The root of curvature in gravitation theory lies in the circumstance that the metric implicitely represents the gravitation field and that the latter (and therefore also the former) will not be constant in the presence of a 'material' energy-momentum source. The roots of non-linearity, that is of self-interaction of the gravitation field, lie in all the factors $\eta^{\alpha\beta}$ of terms in derivatives of the field $h_{\alpha\beta}$ in the flat gravitation Lagrangian (or in the flat gravitation generator). I am afraid that various flat perturbation approaches do not completely account for all self-interaction represented by these factors. As soon as all the pro- and contra-arguments are no longer a matter of principle, but only of a practicaly choice between the one or the other approach in order to get the answer on a definite problem in the easiest and clearest way, one might feel to have understood the opening question.

But even then, there remain other aspects where one might feel the desire for a clearer understanding. The hybrid splitting for instance of $g_{\alpha\beta}$ into $\eta_{\alpha\beta}$ and $\omega\kappa h_{\alpha\beta}$ in (4.1.8) is non-unique and in fact highly arbitrary. From the curved point of view it leads to the truncation of the curved variation (4.1.12) in (4.1.16) into the flat variation in (2.2.2) and to various inconsistencies in the flat theory. Whenever $\omega\kappa h_{\alpha\beta}$ represents a gravitation field coupled to (that means generated by and acting on) an energy-momentum or whatever other quantity, it would seem more satisfactory if similarly $\eta_{\alpha\beta}$ would represent some 'background' field also coupled to that quantity or to some other 'background' quantity. In other words it might seem more meaningful if a deeper and more fundamental significance could be attributed to the unsplit $g_{\alpha\beta}$ coupled in (sometimes non-linear) mathematical expressions to 'external' 'material' and 'internal' gravitation quantities or forms. But if that had some sense, it might perhaps require a very precise and lucid realisation of the vague cosmological idea that a flat $\eta_{\alpha\beta}$ should represent some 'background' gravitation field generated by the 'background matter' in the whole universe, such as may be suggested by the as yet obscure and speculative, but possibly profound principle of Mach.

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